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Symmetries and constant mean curvature surfaces

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Abstract

In this paper, we discuss the Lie symmetries, symmetry algebra and symmetry reductions of the equation which describes constant mean curvature surfaces via the generalized Weierstrass–Enneper formulae. First we point out that the equation admits an infinite-dimensional symmetry Lie algebra. Then using symmetry reductions, we obtain two integrable Hamiltonian systems (one autonomous, the other nonautonomous) with two degrees of freedom. The autonomous one was obtained by Konopelchenko and Taimanov by other means. Our method provides a new approach for construction of constant mean curvature surfaces.

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1. Introduction

For a long time, the only known examples of constant mean curvature surfaces in three-dimensional Euclidean space E^3 were, besides the round sphere and the cylinder, a family of rotationally invariant surfaces discovered in 1841 by Delaunay [1]. But in the past two decades, there have been several breakthroughs in the study of constant mean curvature surfaces. On the one hand, in 1984 Wente [2] constructed infinitely many immersed tori of constant mean curvature and disproved the so-called Hopf conjecture. In 1987 Kapouleas [3] constructed closed constant mean curvature surfaces of any genus $g \geqslant 3$. On the other hand, in 1979 Kenmotsu [4] discovered a remarkable representation formula for arbitrary surfaces in E^3 with nonvanishing mean curvature, which is a generalization of the well known Weierstrass–Enneper formula of minimal surfaces. In 1993 Konopelchenko [5] rediscovered it in a different but equivalent form in connection with integrable nonlinear equations. By using this generalized Weierstrass–Enneper formula, [6] established a relationship between constant mean curvature surfaces and an integrable Hamiltonian system with two degrees of freedom. In this paper, we discuss the Lie symmetries, symmetry algebra and symmetry reductions of the equation which describes constant mean curvature surfaces via the generalized Weierstrass–Enneper formula. We use the notation and formulae from [6].

Let $z, \bar{z}$ be local coordinates on a surface and (X^1, X^2, X^3) coordinates of its immersion in E^3, where

$$\begin{aligned} X^1 + \mathrm{i}X^2 &= 2\mathrm{i}\int_{z_0}^{z} (\bar{\psi}_1^2\,\mathrm{d}z' - \bar{\psi}_2^2\,\mathrm{d}\bar{z}') \\ X^1 - \mathrm{i}X^2 &= 2\mathrm{i}\int_{z_0}^{z} (\psi_2^2\,\mathrm{d}z' - \psi_1^2\,\mathrm{d}\bar{z}') \\ X^3 &= -2\int_{z_0}^{z} (\psi_2\bar{\psi}_1\,\mathrm{d}z' + \psi_1\bar{\psi}_2\,\mathrm{d}\bar{z}') \end{aligned} \tag{1}$$

and ψ_1, ψ_2 satisfy the equation

$$\begin{aligned} \psi_{1t} - \mathrm{i}\psi_{1x} &= 2H(|\psi_1|^2 + |\psi_2|^2)\psi_2 \\ \psi_{2t} + \mathrm{i}\psi_{2x} &= -2H(|\psi_1|^2 + |\psi_2|^2)\psi_1 \end{aligned} \tag{2}$$

where $H = \text{constant}$, $z = t + \mathrm{i}x$.

Then (1) describes a surface with constant mean curvature H. In the following sections, we first point out that equation (2) admits an infinite-dimensional symmetry Lie algebra. Then using symmetry reductions we obtain two integrable Hamiltonian systems (one autonomous, the other nonautonomous) with two degrees of freedom. The autonomous one was obtained in [6] by other means.

2. An infinite-dimensional symmetry Lie algebra

Without loss of generality, we may assume $H = \frac{1}{2}$, then (2) becomes

$$\begin{aligned} \psi_{1t} &= \mathrm{i}\psi_{1x} + (|\psi_1|^2 + |\psi_2|^2)\psi_2 \equiv K_1 \\ \psi_{2t} &= -\mathrm{i}\psi_{2x} - (|\psi_1|^2 + |\psi_2|^2)\psi_1 \equiv K_2. \end{aligned} \tag{3}$$

The Gateaux derivatives of K_i with respect to ψ_j are, respectively,

$$\begin{aligned} K'_{1\psi_1} &= \mathrm{i}\partial_x + \bar{\psi}_1\psi_2 + \psi_1\psi_2\hbar \\ K'_{1\psi_2} &= |\psi_1|^2 + 2|\psi_2|^2 + \psi_2^2\hbar \\ K'_{2\psi_1} &= -|\psi_2|^2 - 2|\psi_1|^2 - \psi_1^2\hbar \\ K'_{2\psi_2} &= -\mathrm{i}\partial_x - \psi_1\bar{\psi}_2 - \psi_1\psi_2\hbar \end{aligned} \tag{4}$$

where operators ∂_x and $\hbar$ are defined, respectively, by

$$\partial_x(f) = \frac{\partial f}{\partial x} \qquad \hbar(f) = \bar{f}$$

for an arbitrary complex-valued function f, the bar denotes the complex conjugation.

$\binom{\sigma_1}{\sigma_2}$ is called a symmetry of equation (3) if it satisfies

$$\begin{pmatrix}\sigma_1 \\ \sigma_2\end{pmatrix}_t = \begin{pmatrix} K'_{1\psi_1} & K'_{1\psi_2} \\ K'_{2\psi_1} & K'_{2\psi_2} \end{pmatrix} \begin{pmatrix}\sigma_1 \\ \sigma_2\end{pmatrix} \tag{5}$$

where σ_{it} denote the total derivatives of σ_i with respect to t, $K'_{i\psi_j}$ the Gateaux derivatives of K_i with respect to ψ_j, and ψ_j satisfy equation (3).

From (5), through an arduous calculation, we have the following theorem.

Theorem 1. *Let*

$$\begin{aligned} \sigma_1 &= a(x,t)\psi_{1x} + b(x,t)\psi_{1t} + c(x,t)\psi_1 \\ \sigma_2 &= a(x,t)\psi_{2x} + b(x,t)\psi_{2t} + d(x,t)\psi_2 \end{aligned} \tag{6}$$

where $a(x,t)$, $b(x,t)$ are real-valued functions, $c(x,t)$, $d(x,t)$ are complex-valued functions, and satisfy the following conditions:

$$\begin{array}{lll} a_t = -b_x & a_x = b_t & b_t - \mathrm{i}b_x = \bar{c} + d \\ c_t = \mathrm{i}c_x & d_t = -\mathrm{i}d_x & c + \bar{c} = d + \bar{d} \end{array} \tag{7}$$

then $\binom{\sigma_1}{\sigma_2}$ *is a symmetry of equation (3).*

Examples.

$$\begin{pmatrix}\psi_{1x}\\ \psi_{2x}\end{pmatrix} \qquad \begin{pmatrix}\psi_{1t}\\ \psi_{2t}\end{pmatrix} \qquad \begin{pmatrix}\mathrm{i}\psi_1\\ \mathrm{i}\psi_2\end{pmatrix}$$

are all symmetries of equation (3) and so is

$$\begin{pmatrix} t\psi_{1x} - x\psi_{1t} - \frac{1}{2}\mathrm{i}\psi_1 \\ t\psi_{2x} - x\psi_{2t} + \frac{1}{2}\mathrm{i}\psi_2 \end{pmatrix}.$$

Suppose $\binom{\sigma_{1j}}{\sigma_{2j}}$ $(j = 1, 2)$ are two symmetries of equation (3), where

$$\begin{aligned} \sigma_{1j} &= a_j\psi_{1x} + b_j\psi_{1t} + c_j\psi_1 \\ \sigma_{2j} &= a_j\psi_{2x} + b_j\psi_{2t} + d_j\psi_2 \end{aligned} \tag{8}$$

and

$$\begin{array}{lll} a_{jt} = -b_{jx} & a_{jx} = b_{jt} & b_{jt} - \mathrm{i}b_{jx} = \bar{c}_j + d_j \\ c_{jt} = \mathrm{i}c_{jx} & d_{jt} = -\mathrm{i}d_{jx} & c_j + \bar{c}_j = d_j + \bar{d}_j. \end{array} \tag{9}$$

Defining the Lie bracket of $\binom{\sigma_{11}}{\sigma_{21}}$ and $\binom{\sigma_{12}}{\sigma_{22}}$ as follows:

$$\left[\begin{pmatrix}\sigma_{11}\\ \sigma_{21}\end{pmatrix}, \begin{pmatrix}\sigma_{12}\\ \sigma_{22}\end{pmatrix}\right] = \begin{pmatrix}\sigma_{11}\\ \sigma_{21}\end{pmatrix}'\begin{pmatrix}\sigma_{12}\\ \sigma_{22}\end{pmatrix} - \begin{pmatrix}\sigma_{12}\\ \sigma_{22}\end{pmatrix}'\begin{pmatrix}\sigma_{11}\\ \sigma_{21}\end{pmatrix} \tag{10}$$

where

$$\begin{pmatrix}\sigma_{1j}\\ \sigma_{2j}\end{pmatrix}' = \begin{pmatrix} a_j\partial_x + b_j\partial_t + c_j & 0 \\ 0 & a_j\partial_x + b_j\partial_t + d_j \end{pmatrix}$$

then we have:

Theorem 2. *The symmetries of equation (3) in theorem 1 together with the Lie bracket (10) constitute an infinite-dimensional Lie algebra over real domain.*

Proof. It is obvious that the real coefficients linear combinations of two symmetries of equation (3) are also symmetries of (3).

If (8) are two symmetries of equation (3), then we have

$$\left[\begin{pmatrix}\sigma_{11}\\ \sigma_{21}\end{pmatrix}, \begin{pmatrix}\sigma_{12}\\ \sigma_{22}\end{pmatrix}\right] = \begin{pmatrix} a\psi_{1x} + b\psi_{1t} + c\psi_1 \\ a\psi_{2x} + b\psi_{2t} + d\psi_2 \end{pmatrix}$$

where

$$\begin{aligned} a(x,t) &= a_1a_{2x} - a_2a_{1x} + b_1a_{2t} - b_2a_{1t} \\ b(x,t) &= a_1b_{2x} - a_2b_{1x} + b_1b_{2t} - b_2b_{1t} \\ c(x,t) &= a_1c_{2x} - a_2a_{1x} + b_1c_{2t} - b_2c_{1t} \\ d(x,t) &= a_1d_{2x} - a_2d_{1x} + b_1d_{2t} - b_2d_{1t}. \end{aligned}$$

Therefore

$$\begin{array}{lll} a_t = -b_x & a_x = b_t & b_t - \mathrm{i}b_x = \bar{c} + d \\ c_t = \mathrm{i}c_x & d_t = -\mathrm{i}d_x & c + \bar{c} = d + \bar{d}. \end{array}$$

From theorem 1, $[\binom{\sigma_{11}}{\sigma_{21}}, \binom{\sigma_{12}}{\sigma_{22}}]$ is also a symmetry of (3).

Therefore from proposition 1 of [7, p 210], theorem 2 is proved. □

3. Symmetry reductions and integrable Hamiltonian systems

In this section, using symmetry reductions, we give two integrable Hamiltonian systems with two degrees of freedom. One is autonomous, which was obtained in [6] by other means. The other is nonautonomous. In a special case, we give explicit solutions in terms of an elliptic function.

First

$$\begin{pmatrix} \sigma_1 = \psi_{1x} - \mathrm{i}\lambda\psi_1 \\ \sigma_2 = \psi_{2x} - \mathrm{i}\lambda\psi_2 \end{pmatrix} \tag{11}$$

is a symmetry of equation (3), where λ is an arbitrary real number.

Let

$$\sigma_1 = \sigma_2 = 0.$$

We have

$$\psi_1 = \phi_1(t)\mathrm{e}^{\mathrm{i}\lambda x} \qquad \psi_2 = \phi_2(t)\mathrm{e}^{\mathrm{i}\lambda x} \tag{12}$$

where $\phi_1(t)$ and $\phi_2(t)$ are arbitrary functions of t.

Then from (3), $\phi_1(t)$ and $\phi_2(t)$ satisfy the following equation:

$$\begin{aligned} \phi_{1t} &= -\lambda\phi_1 + (|\phi_1|^2 + |\phi_2|^2)\phi_2 \\ \phi_{2t} &= \lambda\phi_2 - (|\phi_1|^2 + |\phi_2|^2)\phi_1. \end{aligned} \tag{13}$$

This is an integrable autonomous Hamiltonian system obtained in [6].

Second

$$\begin{pmatrix} \sigma_1 = t\psi_{1x} - x\psi_{1t} \\ \sigma_2 = t\psi_{2x} - x\psi_{2t} + \mathrm{i}\psi_2 \end{pmatrix} \tag{14}$$

is also a symmetry of equation (3). Let

$$\sigma_1 = \sigma_2 = 0.$$

We have

$$\psi_1 = \phi_1(\xi) \qquad \psi_2 = \phi_2(\xi)\mathrm{e}^{-\mathrm{i}\theta} \tag{15}$$

where $\xi = x^2 + t^2$, $\theta = \arctan\frac{x}{t}$, ϕ_1 and ϕ_2 are arbitrary complex-valued functions of ξ. Then from (3), ϕ_1 and ϕ_2 satisfy the following equation:

$$\begin{aligned} 2\xi^{\frac{1}{2}}\phi_1' &= (|\phi_1|^2 + |\phi_2|^2)\phi_2 \\ 2\xi^{\frac{1}{2}}\phi_2' + \xi^{-\frac{1}{2}}\phi_2 &= -(|\phi_1|^2 + |\phi_2|^2)\phi_1. \end{aligned} \tag{16}$$

Let $\phi_1 = p_1 + \mathrm{i}p_2$, $\phi_2 = q_1 + \mathrm{i}q_2$, then

$$\begin{aligned} 2\xi^{\frac{1}{2}}p_j' &= (p_1^2 + p_2^2 + q_1^2 + q_2^2)q_j \\ 2(\xi^{\frac{1}{2}}q_j)' &= -(p_1^2 + p_2^2 + q_1^2 + q_2^2)p_j. \end{aligned} \tag{17}$$

Introduce new variables

$$\mathcal{P}_j = p_j \qquad \mathcal{Q}_j = \xi^{\frac{1}{2}}q_j.$$

Then we have

$$\begin{aligned} \mathcal{P}_j' &= \tfrac{1}{2}(\mathcal{P}_1^2 + \mathcal{P}_2^2 + \xi^{-1}\mathcal{Q}_1^2 + \xi^{-1}\mathcal{Q}_2^2)\xi^{-1}\mathcal{Q}_j \\ \mathcal{Q}_j' &= -\tfrac{1}{2}(\mathcal{P}_1^2 + \mathcal{P}_2^2 + \xi^{-1}\mathcal{Q}_1^2 + \xi^{-1}\mathcal{Q}_2^2)\mathcal{P}_j. \end{aligned} \tag{18}$$

It has the Hamiltonian form

$$\mathcal{P}_j' = \{\mathcal{P}_j, \mathcal{H}\} \qquad \mathcal{Q}_j' = -\{\mathcal{Q}_j, \mathcal{H}\} \tag{19}$$

with the Hamiltonian function

$$\mathcal{H} = \tfrac{1}{8}(\mathcal{P}_1^2 + \mathcal{P}_2^2 + \xi^{-1}\mathcal{Q}_1^2 + \xi^{-1}\mathcal{Q}_2^2)^2 \tag{20}$$

and with respect to the usual Poisson bracket $\{\cdot, \cdot\}$ on Euclidean space E^4. It must be pointed out that because of the nonautonomy of (19), $\mathcal{H}$ is not a first integral for it. In fact, a function $I(x, t)$ is a first integral for (19) if and only if [8]

$$\frac{\partial I}{\partial t} + \{I, \mathcal{H}\} = 0$$

for all x, t.

The Hamiltonian system (19) has two first integrals:

$$I_1 = \xi(\mathcal{P}_1^2 + \mathcal{P}_2^2 + \xi^{-1}\mathcal{Q}_1^2 + \xi^{-1}\mathcal{Q}_2^2)^2 + 2(\mathcal{P}_1\mathcal{Q}_1 + \mathcal{P}_2\mathcal{Q}_2) \tag{21}$$

$$I_2 = \mathcal{P}_1\mathcal{Q}_2 - \mathcal{P}_2\mathcal{Q}_1. \tag{22}$$

Moreover, they are in involution; thus we conclude that the Hamiltonian system (19) is integrable [9].

For the case $p_2 \equiv q_2 \equiv 0$, we can give explicit solutions of (17) in terms of an elliptic function. In fact, let

$$p_1 = p(\xi) \qquad q_1 = q(\xi) \qquad p_2 \equiv q_2 \equiv 0.$$

Then

$$\begin{aligned} 2\xi^{\frac{1}{2}} p' &= (p^2 + q^2)q \\ 2\xi^{\frac{1}{2}} q' &= -(p^2 + q^2)p - \xi^{-\frac{1}{2}}q \end{aligned} \tag{23}$$

and

$$I = \xi(p^2 + q^2)^2 + 2\xi^{\frac{1}{2}} pq \tag{24}$$

is a first integral for (23).

Let

$$p = r\cos\tau \qquad q = r\sin\tau.$$

Then we have

$$\begin{aligned} r' &= -\tfrac{1}{2}\xi^{-1} r\sin^2\tau \\ \tau' &= -\tfrac{1}{2}\xi^{-\frac{1}{2}} r^2 - \tfrac{1}{2}\xi^{-1}\sin\tau\cos\tau. \end{aligned} \tag{25}$$

From (24)

$$r^2 = \frac{-\sin(2\tau) + \sqrt{4I + \sin^2(2\tau)}}{2\xi^{\frac{1}{2}}.}$$

Therefore

$$\begin{aligned} \tau' &= -\tfrac{1}{4}\xi^{-1}\sqrt{4I + \sin^2(2\tau)} \\ \ln r &= -\frac{1}{2}\int \frac{\sin^2\tau}{\xi}\,\mathrm{d}\xi. \end{aligned} \tag{26}$$

The Gaussian curvatures of the corresponding surfaces are [6]

$$K = -\frac{1}{4}\frac{\Delta(\ln r^2)}{r^4}$$

where Δ is the Laplace operator:

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial t^2}.$$

Then from (26), we have

$$K = \frac{-\sin(2\tau)\sqrt{4I + \sin^2(2\tau)}}{\left(-\sin(2\tau) + \sqrt{4I + \sin^2(2\tau)}\right)^2}.$$

The corresponding surfaces are the well known Delaunay surfaces.

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